MATHAI_31_HL_Summer_2021_Q1

Solution

To find the shortest time from sunrise to sunset at point A on Mars, we follow a systemic derivation based on the provided orbital and rotational models for the planet.

1. Verification of the Orbital Parameter b The Martian year is given as 669 Martian days. The function $R(t)$ represents a cyclic process over one Martian year, implying its **period** T is 669.

$$T = \frac{2\pi}{b}$$
$$669 = \frac{2\pi}{b}$$
$$b = \frac{2\pi}{669} \approx 0.00939189...$$

Thus, $b \approx 0.00939$.

2. Calculation of Rotational Velocity Mars completes a full rotation (2π radians) in 24 hours and 40 minutes.

- Conversion to total hours: $24 + \frac{40}{60} = \frac{74}{3}$ hours.
- Hourly rotation angle ω_{hour}:

$$\omega_{\text{hour}} = \frac{2\pi}{74/3} = \frac{6\pi}{74} = \frac{3\pi}{37} \approx 0.25472 \quad \text{rad/h}$$

3. Analysis of Sunrise Moment at Point A The moment of sunrise depends on the axial tilt δ, where $\delta \in [-0.440, 0.440]$. The angle ω is given by $\cos\omega = 0.839\tan\delta$.

- For Maximum ω: $\delta = -0.440$ (since $\cos\omega$ is decreasing on $[0,\pi]$ and $\tan\delta$ is increasing). $\cos\omega = 0.839\tan(-0.440) \approx -0.3951...$ $\omega = \arccos(-0.3951...) \approx 1.976... \approx 1.98$.
- For Minimum ω: $\delta = 0.440$. $\cos\omega = 0.839\tan(0.440) \approx 0.3951...$ $\omega = \arccos(0.3951...) \approx 1.165... \approx 1.17$.

4. Determining Maximum and Minimum of $R(t)$ $R(t)$ is the time (in hours) from the start of the day to sunrise.

$$R(t)_{\text{max}} = \frac{\omega_{\text{max}}}{\omega_{\text{hour}}} = \frac{1.9763...}{0.25472...} \approx 7.758... \quad \text{hours}$$
$$R(t)_{\text{min}} = \frac{\omega_{\text{min}}}{\omega_{\text{hour}}} = \frac{1.1652...}{0.25472...} \approx 4.574... \quad \text{hours}$$

5. Parameters for the Sunrise Model $R(t)$ The function is $R(t) = a\sin(bt) + c$.

- Amplitude a:

$$a = \frac{R_{\max} - R_{\min}}{2}$$
$$= \frac{7.7584 - 4.5745}{2} = 1.59195 \approx 1.6$$

- Vertical shift c:

$$c = \frac{R_{\max} + R_{\min}}{2}$$
$$= \frac{7.7584 + 4.5745}{2} = 6.16645 \approx 6.17$$

6. Determining the Length of the Martian Day $L(t)$ The length of time until sunset is $S(t) = 1.5\sin(0.00939t + 2.83) + 18.65$. The length of day $L(t)$ is the difference between sunset and sunrise:

$$L(t) = S(t) - R(t)$$

Given $L(t) = 1.5\sin(0.00939t + 2.83) - 1.6\sin(0.00939t) + d$:

$$L(t) = [1.5\sin(0.00939t + 2.83) + 18.65] - [1.6\sin(0.00939t) + 6.17]$$
$$d = 18.65 - 6.17 = 12.48$$

7. Shortest Time From Sunrise to Sunset We use **phasor addition** to find the amplitude of $f(t) = 1.5\sin(bt + 2.83) - 1.6\sin(bt)$. Using $z_1 = 1.5e^{i(bt+2.83)}$ and $z_2 = 1.6e^{ibt}$:

$$f(t) = \text{Im}\ (z_1 - z_2) = \text{Im}\ (e^{ibt}(1.5e^{i2.83} - 1.6))$$
$$1.5e^{i2.83} - 1.6 = 1.5(\cos 2.83 + i\sin 2.83) - 1.6$$
$$\approx 1.5(-0.951 + i0.306) - 1.6$$
$$\approx -3.027 + i0.459$$

The modulus p is $\sqrt{(-3.027)^2 + (0.459)^2} \approx 3.061$. The function $L(t)$ is then $L(t) = 3.061\sin(0.00939t + r) + 12.48$. The minimum value (shortest time) occurs when the sine term is -1:

$$L_{\min} = 12.48 - 3.061$$
$$= 9.419 \text{ hours}$$

$$\boxed{9.42 \text{ hours}}$$

MATHAI_31_HL_Summer_2021_Q2

Solution

1. Independence Analysis of Interview Ratings

To determine if staying with the firm is independent of the interview rating, we perform a **chi-squared test for independence** on the provided 2×3 contingency table.

- **Hypotheses**
  - H_0: Staying with the firm is independent of the interview rating.
  - H_1: Staying with the firm is not independent of (is associated with) the interview rating.
- **Data Processing** The observed frequencies (O_{ij}) are:

  $$(12 \;\; 20 \;\; 19; 10 \;\; 15 \;\; 24)$$

  Calculating marginal totals:
  - Row totals: $R_1 = 12 + 20 + 19 = 51$, $R_2 = 10 + 15 + 24 = 49$
  - Column totals: $C_1 = 12 + 10 = 22$, $C_2 = 20 + 15 = 35$, $C_3 = 19 + 24 = 43$
  - Grand total: $N = 100$
- **Test Statistics** The expected frequency for each cell is $E_{ij} = \frac{R_i \times C_j}{N}$:

  $$E_{1,1} = 11.22, \quad E_{1,2} = 17.85, \quad E_{1,3} = 21.93$$
  $$E_{2,1} = 10.78, \quad E_{2,2} = 17.15, \quad E_{2,3} = 21.07$$

  The test statistic value is $\chi^2 = 1.2587...$ and the **degrees of freedom** $df = (2-1)(3-1) = 2$. Using the **p-value** approach at the 5% level ($\alpha = 0.05$):

  $$p \text{ -value} = 0.5329...$$

- **Conclusion** Since $0.533 > 0.05$, we fail to reject H_0. There is insufficient evidence to suggest an association between interview ratings and employee retention.

2. Stratified Sampling

The firm uses **stratified sampling** proportional to the size of the departments.

- Total new employees: $N = 91$
- National department size: $N_{\text{nat}} = 55$
- Target sample size: $n = 18$

The number of employees selected from the national department is:

$$\begin{aligned} n_{\text{nat}} &= \frac{N_{\text{nat}}}{N} \times n \\ &= \frac{55}{91} \times 18 \\ &= 10.879... \approx 11 \end{aligned}$$

3. Correlation and Reliability Analysis

- **Appropriateness of Correlation (c-i)**: Calculating a single correlation coefficient for all 18 employees may be inappropriate because the data consists of two distinct groups (National and International departments). The scatter diagram suggests different underlying distributions or trends for each group (a **bimodal distribution** or “clustering”), which can distort the overall correlation.
- **Reliability Test (d-i)**: This is a **test-retest reliability** study, often assessed using **Pearson product-moment correlation** or **Spearman’s rank correlation** between the first and second marks.
- **Hypothesis Test (d-ii)**:
  - $H_0 : \rho = 0$, $H_1 : \rho > 0$.
  - Sample size $n = 7$.
  - Using the data for employees L through R, the calculated correlation coefficient is $r \approx 0.816$.
  - For $n = 7$ at $\alpha = 0.05$, the critical value for a one-tailed test is approximately 0.669.
  - Since $0.816 > 0.669$, we reject H_0.
- **Reliability Comment (d-iii)**: There is a significant positive correlation between the first and second marks, indicating that the written assessment is **reliable**.

4. Multiple Hypothesis Testing

- **Number of Tests (e-i)**: With 5 sections and 5 attributes, the firm compares every pair.

$$\text{Number of tests} = 5 \times 5 = 25$$

- **Probability of Type I Error (e-ii)**: Under the assumption of independence and no actual correlation, find the probability of at least one significant result ($\alpha = 0.05$).

$$\begin{aligned} P(\text{at least one significant}) &= 1 - P(\text{none significant}) \\ &= 1 - (1 - 0.05)^{25} \\ &= 1 - (0.95)^{25} \\ &\approx 0.723 \end{aligned}$$

- **Interpretation (e-iii)**: Given the high probability (72.3%) of obtaining at least one significant result by pure chance when performing 25 tests, a single significant result between section 2 and attribute X should be treated with caution. It may be a **Type I error** (a **false positive**) rather than a meaningful relationship.

Final Answers:

1. (a) H_0: Independent; H_1: Not independent; p -value ≈ 0.533. Conclusion: Fail to reject H_0.
2. (b) Calculation: $\frac{55}{91} \times 18 \approx 10.88 \to 11$.
3. (d-i) Test-retest reliability. (d-ii) $r \approx 0.816 > \text{crit}$; reject H_0. (e-i) 25. (e-ii) $\boxed{0.723}$.